Finite sets

Def: A set A is <u>finite</u> if it's empty or there is a bijection

$$f : A \to \{1, 2, \ldots, n\} \text{ for some } n \in \mathbb{Z}_+$$

In the former case, A has <u>cardinality</u> 0. In the latter, A has <u>cardinality</u> n.

Question: Why is cardinality uniquely determined? Why can't there be $m \neq n$ s.t.

$$f : A \to \{1, \ldots, m\} \text{ and } g : A \to \{1, \ldots, n\} \text{ are both bijections?}$$

We need to rule out this possibility.

Lemma: Let $n \in \mathbb{N} (= \mathbb{Z}_+)$. Let A be a set. If $a \in A$, then $\exists$ a bijetion $f : A \to \{1, \ldots, n+1\} \iff \exists$ a bijection $g : A \to \{1, \ldots, n\}$.

Proof: Suppose $\exists$ a bijection $f : A \to \{1, \ldots, n+1\}$.

Let $f(a) = m$, and let $b \in A$ be the unique elt s.t. $f(b) = n+1$.

If $b = a$, then define $g : A \to \{1, \ldots, n\}$ to be $g = f|_{A - \{a\}}$

If $b \neq a$, define g as follows

$$g(x) = \begin{cases} f(x) & \text{if } x \neq b \\ f(a) & \text{if } x = b \end{cases}$$

(In each of these cases, do you see why g is a bijection?)

Now assume $g: A - \{a\} \to \{1, \dots, n\}$ is a bijection. Define

$$f: A \to \{1, \dots, n+1\} \text{ to be } f(x) = \begin{cases} g(x) & \text{if } x \neq a \\ n+1 & \text{if } x = a \end{cases}$$

(Why is this a bijection?) $\square$

Before we can prove the next theorem, we need mathematical induction.

Axiom of induction: If $P(1)$ is true, and $(P(k) \Rightarrow P(k+1))$ then $P(n)$ holds $\forall\ n \in \mathbb{N}$ (can also start w/ $P(a)$ for $a \in \mathbb{Z}$, and prove $\forall\ n \in \mathbb{Z}$ s.t. $n \geq a$.

Structure of proof:

We want to show $P(n)$ holds for all $n \in \mathbb{N}$.

Base case: Prove $P(1)$.

Induction hypothesis: Assume $P(k)$ holds.

Conclude $P(k+1)$.

Example: We will show by induction that $1 + 2 + \dots + n = \frac{n(n+1)}{2}$ $\forall\ n \in \mathbb{N}$

Base case: If $n = 1$, then $1 + \dots + n = 1 = \frac{1 \cdot 2}{2} = \frac{n(n+1)}{2}$

Induction hypothesis: Assume the equality holds for $n = k$.

Then $1 + 2 + \dots + k + (k+1) = \frac{k(k+1)}{2} + k + 1 = \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$

$$= \frac{(k+1)(k+2)}{2} = \frac{(k+1)((k+1)+1)}{2},$$ so it holds for $k+1$, so it holds $\forall\ n \in \mathbb{N}$, by induction. $\square$

Equivalently...

Strong Induction: If $P(1)$ and $\big((P(j)\ \forall\ j<k+1) \Rightarrow P(k+1)\big)$, then $P(n)\ \forall\ n \in \mathbb{N}$.

Back to finite sets...

Theorem: Let A be a set and suppose $\exists$ a bijection $f: A \to \{1,\dots,n\}$ for some $n \in \mathbb{N}$. Let $B \subsetneq A$ (i.e. B a proper subset of A). Then there is no bijection $g: B \to \{1,\dots,n\}$ and if $B \neq \emptyset$, there exists a bijection $h: B \to \{1,\dots,m\}$ for some $m<n$.

Proof: We prove the theorem by induction on n.

Base case: If $n=1$, then the only proper subset of A is $\emptyset$, which clearly has no bijection w/ $\{1\}$ (why?).

Assume the theorem holds for $n=k$.

Suppose $\exists$ a bijection $f: A \to \{1,\dots,k+1\}$. Let $\underset{\neq\emptyset}{B} \subsetneq A$ be a proper subset.

Let $b \in B$ and $a \in A-B$.

Then $B-\{b\} \subsetneq A-\{b\}$, since $a \in A-\{b\}$, but $a \notin B-\{b\}$.

By the lemma, $\exists$ a bijection $f_0: A-\{b\} \to \{1,\dots,k\}$.

Thus, by the induction hypothesis, we conclude

(1) There is no bijection $g_0: B-\{b\} \longrightarrow \{1,\dots,k\}$, and

(2) $B-\{b\} = \emptyset$ or $\exists$ a bijection $h_0: B-\{b\} \longrightarrow \{1,\dots,m\}$ for $m<k$.

Combining (1) with the lemma, we conclude there is no bijection $g: B \longrightarrow \{1,\dots,k+1\}$.

If $B-\{b\} = \emptyset$, there is a bijection of B w/ $\{1\}$ and $1<k+1$.

Otherwise, $\exists$ a bijection $h_0: B-\{b\} \longrightarrow \{1,\dots,m\}$ for $m<k$, and the lemma implies $\exists$ a bijection $h: B \to \{1,\dots,m+1\}$ and $m+1<k+1$. $\square$

Corollary: If A is finite, there is no bijection of A w/ a proper subset of itself.

Cor: $\mathbb{Z}_+$ is not finite.

Cor: The cardinality of a finite set A is uniquely determined by A.